INVERSE ELASTIC-PLASTIC PROBLEM FOR ANTIPLANE STRAIN

(OBRATNAIA UPRUGO-PLASTICHESKAIA ZADACHA V USLOVIIAKH ANTIPLOSKOI DEFORMATSII)

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By the antiplane strain we understand the state of stress in a cylindrical body of infinite height, produced by a loading acting along the generators of the cylinder and constant along these generators. Suppose that two cylindrical bodies one of which is perfectly rigid, are in contact so that on the surface of contact the adhesion conditions take place. The perfectly rigid body subjected to certain forces is called the punch, and the problem of determination of stresses and strains in the elastic-plastic body obeying the Prandtl diagram, is called the elastic-plastic contact problem. In [1] the elastic-plastic problem for antiplane strain was investigated, assuming that on the boundary of the body tractions were prescribed; consequently, in the plastic region the problem was statically determinate. The elastic-plastic contact problem is statically indeterminate.

L.A. Galin formulated the problem of determination of the contour of the body (or its sections) so that the plastic regions are developed at once on the whole contour (or its sections). Below we examine a number of elastic-plastic contact problems in this inverse formulation (Sections 2,3). In Section 4 we present the solution of the problem considered in [1], for the case when the plastic region is developed at once on a section of the boundary, degenerating into a line on the boundary of the body.

1. Fundamental relations. 1. The fields of displacement and stress in the considered body satisfy the relations

$$u = v = 0, \qquad w = w(x, y), \qquad \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$
$$\tau_{xz} = \tau_{xz}(x, y), \qquad \tau_{yz} = \tau_{yz}(x, y) \tag{1.1}$$

Here u, v, w are the components of the displacement vector,

σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz} are the components of the stress tensor, x, y, z are the Cartesian coordinates (the z-axis is parallel to the generator). In the plastic region we have the following yield condition

$$|\tau| = k, \qquad \tau = \tau_{xz} + i\tau_{yz} = ke^{i\theta} \qquad (1.2)$$

where k is the plasticity constant.

Along the straight line $y = -x \cot \theta + C$, θ = const coinciding with the slip line and orthogonal to vector τ at all points, we have the relation [1]

$$w = \text{const} \qquad (1.3)$$

On the boundary between the elastic and plastic regions we assume the continuity of the stresses and displacements.

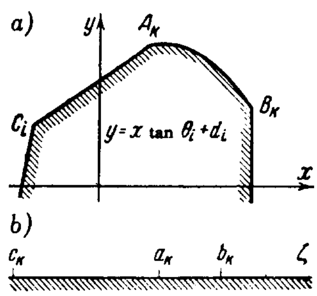

Fig. 1.

In the elastic region the stresses and the displacement can be represented [2] by one analytic complex variable function (μ is the shear modulus)

$$w = \operatorname{Re} f(z), \qquad \tau = \tau_{xz} + i\tau_{yz} = \mu\overline{f'(z)} \qquad (z = x + iy) \qquad (1.4)$$

2. Suppose that the plastic region has a non-zero area and the slip lines emanating from the contour of the body on which the conjunction condition with the punch w = const is given, intersect the boundary between the elastic and plastic regions on a section L. Then it follows from formulas (1.3) and (1.4) that on section L the tangential component of the stress vector τ should vanish, i.e. the slip lines should be tangent to the contour L. If the contour of the body is a convex smooth arc it is impossible to construct a convex smooth contour L resting on this arc and possessing the above property. Apparently in this case the solution in the plastic region is discontinuous.

2. Inverse contact elastic-plastic problem. Suppose that the contour of the body consists of known sections of straight line, which are free of tractions, and curved lines constituting the surfaces of contact with the punches. It is required to determine the curved arcs by means of the condition that the plastic regions rest on the latter. Let A_k, B_k, C_i be the apexes of the polygon constituting the contour of the body (see Fig. 1); A_k, B_k are points of the curved arcs; some of them may be at infinity: $i = 1, \ldots, n$, $k = 1, \ldots, m$. The equations of the rectilinear sections which are free of tractions have the form

$$y = x \tan \theta_j + d_j \qquad (j = 1, \ldots, m + n)$$

where θ_j is the angle between the jth line and the x-axis. The condition of absence of the loading on the jth rectilinear portion of the boundary can, by means of representation (1.4), be written in the form

$$\operatorname{Re}\left[(\tan\theta_j - i) f'(z)\right] = 0 \tag{2.1}$$

On the unknown part of the contour of the body the adhesion condition (1.3) and condition (1.2) should be satisfied; on the basis of formula (1.4) the latter conditions can be written in the form

$$\operatorname{Re} f(z) = h_k, \qquad |f'(z)| = k/\mu \tag{2.2}$$

where h_k is the constant displacement of the kth punch.

Let us now pass to the parametric plane of the complex variable ζ by means of the conformal mapping $z = \omega(\zeta)$, in such a way that points C_i, A_k, B_k of plane z correspond to the points c_i, a_k, b_k of the real axis on plane ζ, and the elastic region corresponds to the upper semi-plane $\operatorname{Im}\zeta > 0$ (Fig. 1). Introduce the notation $f[\omega(\zeta)] = F(\zeta)$. For the determination of the functions $\omega(\zeta)$ and $F(\zeta)$ analytic in the upper semi-plane $\operatorname{Im}\zeta > 0$, we obtain in view of (2.1), (2.2) and the equations of the straight lines the boundary value problem

$$\left|\frac{F'(\zeta)}{\omega'(\zeta)}\right| = \frac{k}{\mu} \quad \text{on } L, \qquad \operatorname{Re}\left[(\tan\theta_j - i)\frac{F'(\zeta)}{\omega'(\zeta)}\right] = 0 \quad \text{on } M \tag{2.3}$$

$$\operatorname{Re} F(\zeta) = h_k \quad \text{on } L, \qquad \operatorname{Re}\left[(\tan\theta_j + i)\,\omega(\zeta)\right] = -d_j \quad \text{on } M \tag{2.4}$$

Here L are points of the real axis situated between a_k and b_k, while M are the remaining points of the real axis.

The boundary value problem (2.3) belongs to the type of problems investigated in [1]; consequently, $F'(\zeta)/\omega'(\zeta)$ is determined independently of $\omega(\zeta)$. Differentiating the boundary condition (2.4) and making use of the solution of the boundary value problem (2.3), to determine $\omega'(\zeta)$, we arrive at the Hilbert problem for the upper semi-plane; the complete solution of this problem is presented in the monographs of Muskhelishvili [3] and Gakhov [4]. In solving these problems the conditions of equilibrium are employed as in the plane problem of the elasticity theory [5]. In calculating the number of zeros necessary for the solution of the boundary value problem (2.3), it is convenient to use the hydrodynamic analogy [6,1].

2. As an example consider the following problem. Suppose that the contour of the body consists of the radii $\arg z = \pm\,\theta_0$ free of traction and an unknown curvilinear arc rigidly connected with the punch which is subjected to force P.

In this case the curved arc is a circle of radius

$$R = \frac{P}{2k\theta_0} \tag{2.5}$$

and the function $f(z)$ is

$$f(z) = \frac{kR}{\mu} \ln z + C \tag{2.6}$$

C is an unknown constant.

3. Inverse elastic-plastic contact problem. Suppose that a rigid punch bounded by known sections of straight lines and by curved arcs is completely immersed in an infinite elastic-plastic body. The plastic region rests on the curved arcs which are to be determined. On the rectilinear sections of the boundary of the body we have in view of representation (1.4) the condition

$$\operatorname{Re} f(z) = h \tag{3.1}$$

where h is a constant.

We use the notations of the preceding section. On the parametric plane ζ for the determination of functions $z = \omega(\zeta)$ and $F(\zeta) = f[\omega(\zeta)]$ we have as in Section 2 the boundary value problem

$$\operatorname{Re} F(\zeta) = h \quad \text{on} \quad L + M \tag{3.2}$$

$$\left|\frac{\omega'(\zeta)}{F'(\zeta)}\right| = \frac{\mu}{k} \quad \text{on } L, \qquad \operatorname{Re}\left[(1 - i \tan\theta_j)\frac{\omega'(\zeta)}{F'(\zeta)}\right] = 0 \quad \text{on } M \tag{3.3}$$

As an example consider the problem when the whole boundary of the rigid punch subjected to force P is known. In this example the boundary of the punch is the circle of radius

$$R = \frac{P}{2\pi k} \tag{3.4}$$

and the function $f(z)$ is

$$f(z) = \frac{P}{2\pi\mu} \ln z + C \qquad (C = \text{const}) \tag{3.5}$$

4. Elastic-plastic problem for the exterior of a contour consisting of sections of straight and curved lines, free of traction, in the case when the plastic region degenerates into curved boundary curves. 1. We use the notations of [1, Section 4]. For the degenerate problem considered, the boundary value problem for the determination of $\omega'(\zeta)$ is distinct from the corresponding boundary value problem in the nondegenerate case [1]

$$\arg \omega'(\zeta) = \frac{\pi}{2} + \theta \quad \text{on } L, \ \operatorname{Re}\,[(\tan\theta_j + i)\,\omega'(\zeta)] = 0 \text{ on } M \tag{4.1}$$

Here the function $\theta = -\arg F(\zeta)$ is known from the solution of another boundary value problem [1, Section 4, (4.3)].

2. Consider a somewhat more general nonlinear boundary value problem: it is required to determine a function $f(\zeta)$ analytic in the upper semi-plane $\operatorname{Im} \zeta > 0$ in accordance with the condition on the real axis t

$$\arg f(t) = \alpha(t) \quad (t \in L), \qquad \operatorname{Re}[a(t) - ib(t) f(t)] = 0 \quad (a + ib \neq 0, \ t \in M) \quad (4.2)$$

Here $a(t)$, $b(t)$, $\alpha(t)$ are almost everywhere continuous functions which satisfy the Holder condition in the continuity intervals. L and M are sections of the real axis.

Let us construct the canonical function $X(\zeta)$ for the boundary value problem (4.2) in exactly the same way as in [1, formulas (3.2), (3.3)]. Introduce the piecewise analytic function $\Phi(\zeta)$

$$\Phi(\zeta) = \begin{cases} f(\zeta) / X^{+}(\zeta) & \text{as } \operatorname{Im} \zeta > 0 \\ \bar{f}(\zeta) / X^{-}(\zeta) & \text{as } \operatorname{Im} \zeta < 0 \end{cases} \quad (4.3)$$

For the function $\Phi(\zeta)$, analytic in the exterior of cuts L of the real axis, we obtain from the boundary condition (4.2) the homogeneous linear Riemann boundary value problem [3, 4]

$$\Phi^{+}(t) = e^{2i\alpha(t)} \Phi^{-}(t) \qquad (t \in L) \quad (4.4)$$

Note. The solution of the problem in the title of the section can also be derived by using the hydrodynamic analogy.

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